

XIII Convegno Nazionale di Relatività Generale e Fisica della Gravitazione

21-25 September 1998, Monopoli (Bari)

Analytic solution of the Regge-Wheeler differential equation for black hole perturbations in radial coordinate and time domains

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Abstract

An analytic solution of the Regge-Wheeler (RW) equation has been found via the Frobenius method at the regular singularity of the horizon $2M$, in the form of a time and radial coordinate dependent series. The RW partial differential equation, derived from the Einstein field equations, represents the first order perturbations of the Schwarzschild metric. The known solutions are numerical in time domain or approximate and asymptotic for low or high frequencies in Fourier domain. The former is of scarce relevance for comprehension of the geodesic equations for a body in the black hole field, while the latter is mainly useful for the description of the emitted gravitational radiation. Instead a time domain solution is essential for the determination of radiation reaction of the falling particle into the black hole, i.e. the influence of the emitted radiation on the motion of the perturbing mass in the black hole field. To this end, a semi-analytic solution of the inhomogeneous RW equation with the source term (Regge-Wheeler-Zerilli equation) shall be the next development.

Keywords: Partial differential equations (35Q75) Equations of motion (83C10) Black holes (83C57)

1 Introduction

Regge and Wheeler [1] proved the vacuum stability of the Schwarzschild black hole, while Zerilli [2, 3] studied the emitted radiation via the radial wave equation for polar perturbations, which source is a freely falling test mass m towards the black hole of large mass M . Moncrief [4] showed the gauge invariant signif-

ificance of the wave equations.

Past work was concerned on the Fourier analysis of the emitted radiation (for a review see Ruffini [5]) and not on the motion of the particle. Along the lines of radiation emission analysis, more recent work has been mainly developed in Japan (Osaka and Tokio), United States (Salt Lake City and Caltech), while symbolic computing on perturbations has been pursued in Canada (Kingston) and Italy (Roma, La Sapienza, ICRA).

Let us examine the known approaches to solutions of the RW equations: a Fourier antitransform from an approximate frequency solution (where low or high frequencies are neglected, and thus also the corresponding antitransformed and yet unspecified time-parts of the trajectory of the incoming test mass) presents controversial issues of interpretation; a numerical time solution is not adequate for comprehension of the different contributions in the geodesic equations. Therefore, an analytic solution, in time and obviously radial coordinate domains, remains the only way for studying the problem of motion.

Past work was confined to a particle falling in an *unperturbed* Schwarzschild metric, but radiation reaction requires a geodesic path through the emission of radiation, in a *perturbed* Schwarzschild metric.

This novel approach for calculation of the motion and radiation reaction for the two-body problem (body plus particle, the small parameter m/M being the ratio of the masses) is based on the following concept: in the background curvature given by the Schwarzschild geometry rippled by gravitational waves, the geodesic equations insure the presence of radiation reaction also for high velocities and strong field (the expansion parameter being m/M and not v/c or ϕ/c). This concept is applicable to any orbit, but radial fall is of interest due to the non-adiabatic regime (equality of radiation reaction and fall time scales), in which *the particle has to locally and immediately react to the emitted radiation*. The energy balance hypothesis (emitted radiation equal to the variation in the kinetic energy) is only used for the determination of the 4-velocity via the Lagrangian and for the normalization of divergencies; the solution in time domain of the Regge-Wheeler-Zerilli-Moncrief radial wave equation determines the metric tensor expressing the polar perturbations, in terms of which the geodesic equations are written; finally radiation reaction shall be identified by subtraction of the 0th-order terms in the geodesic equations [6, 7].

The long-term aim of this work is the identification of radiation reaction in the trajectory up to the horizon, without the assumption of adiabaticity and with minimal use of the energy balance postulate.

Future developments may include: i) a general method for the determination of the motion of small objects in any orbit; ii) a contribution to the solution of the inhomogeneous 2nd-order equations, which energy-momentum tensor is based on the geodesic equations *with perturbation effects* and yet to be found; iii) a

post-Schwarzschildian formalism.

In this paper we are concerned with the partial fulfillment of the first step, namely the solution of the associated homogeneous equation.

Radiation reaction is a fundamental concept in bodies motion theory, but also has relevant implications on detector's templates since the capture of stars by black holes is a source of gravitational waves.

For a brief critical review of the status of comprehension of radiation reaction and references, please refer to [6, 7].

2 The equation

The RWZM equation for polar perturbations is:

$$\frac{d^2\Psi_l(r, t)}{dr^{*2}} - \frac{d^2\Psi_l(r, t)}{dt^2} - V_l(r)\Psi_l(r, t) = S_l(r, t) \quad (1)$$

where

$$r^* = r + 2M \ln \left(\frac{r}{2M} - 1 \right) \quad (2)$$

is the tortoise coordinate and the potential $V_l(r)$ is:

$$V_l(r) = \left(1 - \frac{2M}{r} \right) \frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2r + 18M^3}{r^3(\lambda r + 3M)^2} \quad (3)$$

Further $\lambda = \frac{1}{2}(l-1)(l+2)$ and $S_l(r, t)$ is the 2^l -pole source component and, for a radially falling particle, is expressed by:

$$S_l = \frac{\left(1 - \frac{2M}{r} \right) k}{(\lambda + 1)(\lambda r + 3M)} \left\{ r \left(1 - \frac{2M}{r} \right)^2 \delta'[r - r_0(t)] - \left[(\lambda + 1) - \frac{M}{r} - \frac{6Mr}{\lambda r + 3M} \right] \delta[r - r_0(t)] \right\} \quad (4)$$

where $k = 4m\sqrt{(2l+1)\pi}$ and $r_0(t)$ (known geodesic equation of motion in *unperturbed Schwarzschild metric*) is the inverse of:

$$t = -4M \left(\frac{r}{2M} \right)^{1/2} - \frac{4M}{3} \left(\frac{r}{2M} \right)^{3/2} - 2M \ln \left[\left(\sqrt{\frac{r}{2M}} - 1 \right) \left(\sqrt{\frac{r}{2M}} + 1 \right)^{-1} \right] \quad (5)$$

Eq. 5 reveals that the time domain spans from $-\infty$ to $+\infty$. The total energy radiated after a given t_0 per l mode is:

$$E = \frac{1}{64\pi} \frac{(l+2)!}{(l-2)!} \int_{t_0}^{\infty} (\dot{\Psi})^2 dt \quad (6)$$

3 The analytic solution

Eq.(1) can be rewritten in terms of t and r^* (tortoise coordinate) only, resulting into a partial differential equation with constant coefficients but solely via an approximate inverse function $r(r^*)$ of eq. (2) and thus resulting into an approximate p.d.e. Further, the solution is most interesting at $r^* = -\infty$ where a detailed analytic study is not possible. In the (t, r) domain instead, eq.(1) becomes:

$$\frac{1}{A^2(r)} \frac{d^2 \Psi_l}{dr^2} - \frac{1-A(r)}{rA^2(r)} \frac{d\Psi_l}{dr} - \frac{d^2 \Psi_l}{dt^2} - V_l(r) \Psi_l = S_l(r, t) \quad (7)$$

where $A(r) = \frac{dr^*}{dr} = \frac{r}{r-2M}$. The initial value problem is well defined (Ψ and $\dot{\Psi}$ are zero, i.e. particle at rest at infinity as the boundary conditions which affirm that the same functions are zero at infinity at all times). The associated homogeneous equation of eq. (7) can be reduced to an ordinary differential equation with the position $\Psi(r, t) = R(r)T(t)$. Dividing eq. (7) for RT it is obtained that (dropping the l notation):

$$\frac{1}{RA^2} \frac{d^2 R}{dr^2} - \frac{1-A}{rRA^2} \frac{dR}{dr} - \frac{1}{T} \frac{d^2 T}{dt^2} - V = 0 \quad (8)$$

The first, second and fourth term are not dependent upon t and form time constants. Thus also the third term must be a constant in time and eq. (8) is autonomous [8]:

$$\frac{1}{T} \frac{d^2 T}{dt^2} = \tau^2 \quad (9)$$

Let us pose $dT/dt = u$ and hence $d^2 T/dt^2 = uu_T$ where the lower subscript indicates a derivative to T . Eq. (9) becomes:

$$\frac{1}{T} uu_T = \tau^2 \quad (10)$$

which solution is $T = e^{\pm\tau}$. Eq. (8) becomes:

$$\frac{1}{RA^2} \frac{d^2 R}{dr^2} - \frac{1-A}{rRA^2} \frac{dR}{dr} - \tau^2 - V = 0 \quad (11)$$

or showing the dependence of $A(r)$ with r :

$$\frac{d^2 R}{dr^2} + \frac{2M}{r(r-2M)} \frac{dR}{dr} - \frac{r^2 R}{(r-2M)^2} (\tau^2 + V) = 0 \quad (12)$$

For searching the solution, it is better to recast eq. (12) as:

$$\frac{d^2 R}{dr^2} + \frac{2M}{r(r-2M)} \frac{dR}{dr} - f(r)R = 0 \quad (13)$$

where $f(r)$ is given by:

$$\frac{\lambda^2 \tau^2 r^6 + 6\lambda \tau^2 M r^5 + (9\tau^2 M^2 + 2\lambda^3 + 2\lambda^2) r^4 + 2\lambda^2 M(1-2\lambda) r^3 + 6\lambda M^2(3-2\lambda) r^2 + 18M^3(1-2\lambda) r - 36M^4}{r^2(r-2M)^2(\lambda r + 3M)^2} \quad (14)$$

With the position $\rho = r - 2M$, eq. (13) is finally recast as:

$$(\rho+2m)^2[\lambda(\rho+2M)+3M]^2 \frac{d^2 R(\rho)}{d\rho^2} + \frac{2M(\rho+2m)[\lambda(\rho+2M)+3M]^2}{\rho} \frac{dR(\rho)}{d\rho} + \frac{\gamma(\rho)}{\rho^2} R(\rho) = 0 \quad (15)$$

where

$$\begin{aligned} \gamma(\rho) = & -[\lambda^2 \tau^2 (\rho+2M)^6 + 6\lambda \tau^2 M (\rho+2M)^5 + (9\tau^2 M^2 + 2\lambda^3 + 2\lambda^2) (\rho+2M)^4 \\ & + 2\lambda^2 M(1-2\lambda) (\rho+2M)^3 + 6\lambda M^2(3-2\lambda) (\rho+2M)^2 + 18M^3(1-2\lambda) (\rho+2M) - 36M^4] \end{aligned} \quad (16)$$

Eq. (15) may be solved with the Frobenius method [9]. Let us rewrite eq. (15) as:

$$\alpha(\rho) \frac{d^2 R(\rho)}{d\rho^2} + \frac{\beta(\rho)}{\rho} \frac{dR(\rho)}{d\rho} + \frac{\gamma(\rho)}{\rho^2} R = 0 \quad (17)$$

where the expressions of $\alpha(\rho)$ and $\beta(\rho)$ are straightforwardly derived from (15). The functions α , β and γ are regular about the expansion point $\rho = 0$ and have the form:

$$\alpha(\rho) = \sum_{n=0}^4 \alpha_n \rho^n \quad \beta(\rho) = \sum_{n=0}^3 \beta_n \rho^n \quad \gamma(\rho) = \sum_{n=0}^6 \gamma_n \rho^n \quad (18)$$

The indicial equation ($\alpha_0 s^2 + (\beta_0 - \alpha_0)s + \gamma_0 = 0$) is:

$$4M^4(2\lambda+3)^2 s^2 + [4M^4(2\lambda+3)^2 - 4M^4(2\lambda+3)^2]s - 16K^2 M^6(2\lambda+3)^2 = 0 \quad (19)$$

and reveals that the roots are $s_{1,2} = \pm 2KM$ and differ by an integer ($4KM$). One solution $R_a(\rho)$ is a series of the form:

$$R_a(\rho) = \rho^s \sum_{n=0}^{\infty} R_0 \rho^n \quad (20)$$

while the other is of the form:

$$R_b(\rho) = \ln \rho \left[\sum_{n=0}^{\infty} (s + 2KM) R_n(s) \rho^{n+s} \right]_{s=-2KM} + \rho^{-2KM} \sum_{n=0}^{\infty} \frac{d}{ds} [(s + 2km) R_n(s)]_{s=-2KM} \rho^n \quad (21)$$

R_0 has an arbitrary value and all other terms depend upon it, e.g.:

$$R_1 = -4 \frac{M^3 [(-16\lambda^2 - 102\lambda - 36)K^2 M^2 + (8\lambda^2 + 12\lambda + 9)KM + (4\lambda^2 + 10\lambda + 9)]}{4KM + 1} R_0 \quad (22)$$

4 Acknowledgements

Thanks are due to NIKHEF (Nationaal Instituut voor Kernfysica en Hoge Energie Fysika) and FOM (stichting voor Fundamenteel Onderzoek der Materie) for their Research Grant in Mathematical Physics at the Theory section of NIKHEF-Amsterdam; to Prof. I.W. Roxburgh of the Astronomy Unit, School of Mathematical Sciences, Queen Mary & Westfield College, London; finally the access to the computing facilities of the Mathematics Dept. "G. Castelnuovo" of Roma I University, La Sapienza is acknowledged.

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